CODIMENSION ONE MINIMAL FOLIATIONS WHOSE LEAVES HAVE FUNDAMENTAL GROUPS WITH THE SAME POLYNOMIAL GROWTH

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ABSTRACT. Let $\mathcal{F}$ be a transversely orientable codimension one minimal foliation without vanishing cycles of a manifold M. We show that if the fundamental group of each leaf of $\mathcal{F}$ has polynomial growth of degree k ($k \in \mathbb{Z}_{\geq 0}$), then the foliation $\mathcal{F}$ is without holonomy.

1. INTRODUCTION

In [YT08], we have considered the following question: how is the foliation without vanishing cycles if the fundamental group of each leaf is isomorphic to an elementary group?

We have showed the following in [YT08]: If the fundamental groups of all leaves are isomorphic to $\mathbb{Z}$, then the foliation is without holonomy. But if the fundamental groups of the leaves are "complicated", then the foliation can have nontrivial holonomy. In this paper, we extend the result as follows.

Theorem 1.1. *Let $\mathcal{F}$ a codimension one transversely orientable C^0 minimal foliation without vanishing cycles of a manifold M. If the fundamental group of each leaf L has polynomial growth of degree k for some nonnegative integer k, then the foliation $\mathcal{F}$ is without holonomy.*

Note that any group which has polynomial growth is finitely generated.

2. PRELIMINARIES

First, we recall a vanishing cycle in the sense of Novikov [N65].

Definition 2.1. Let $\mathcal{F}$ be a foliation of a manifold M. A loop γ on the leaf $L \in F$ is called a vanishing cycle if there is a mapping $F : S^1 \times [0,1] \to M$ such that arcs $F(x, [0,1])$ for every $x \in S^1$ are transverse to $\mathcal{F}$, each loop $F(S^1, t)$ for any $t \in [0,1]$ is contained in some leaf L_t, $[F(S^1, 0)] = [\gamma] \neq 1 \in \pi_1(L)$, and $[F(S^1, s)] = 1 \in \pi_1(L_s)$ for all $s \in]0,1]$.

We define a trivial fence as follows [Y09].

Definition 2.2 (Trivial fences)**.** Let $\mathcal{F}$ be a transversely oriented foliation of a manifold M. For a compact subset K of a leaf L_0 of $\mathcal{F}$, an embedding $F : K \times$

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$[0,1] \longrightarrow M$ is called a positive trivial fence if $F(K \times \{t\})$ is contained in a leaf L_t of $\mathcal{F}$, $F|K \times \{0\}$ is the inclusion $K \subset L_0$ and $F(\{x\} \times [0,1])$ is transverse to $\mathcal{F}$.

Remark 2.3. For an arcwise connected compact subset K of a leaf L, a positive trivial fence on K exists if and only if the holonomy on the positive side along any loop in K is trivial.

To prove the main theorem, the following statement will be helpful.

Lemma 2.4. *Let G be a group with polynomial growth of degree $k \in \mathbb{Z}_{\geq 0}$ and $H \leq G$ subgroup of G with the same growth. Then, for any $g \in G$, there is a positive integer $n \in \mathbb{Z}_{>0}$ such that $g^n \in H$.*

Proof. Let $\mathcal{G}$ be a generating set of G and $\mathcal{H}$ a generating set of H with $\mathcal{H} \subseteq \mathcal{G}$. We define the word lengths on G and H by these generating sets. Let G_m be the set of the elements of G whose word lengths are at most m and H_m the set of the elements of H whose word lengths are at most m. By contradiction, suppose that there exists $g_0 \in G$ such that $g_0^n \notin H$ for any $n \in \mathbb{Z}_{>0}$.

Claim 1. For $n, m \in \mathbb{Z}$ and $h, h' \in H$, $g_0^n h' = g_0^m h$ if and only if $n = m$ and $h' = h$. In particular, $g_0^j H_k \cap g_0^{j'} H_{k'} = \emptyset$ for any $j \neq j'$.

Indeed, if $n = m$ and $h' = h$, then obviously $g_0^n h' = g_0^m h$. Conversely, suppose that $g_0^n h' = g_0^m h$. If $n \neq m$, then $H \ni h'h^{-1} = g_0^{m-n} \notin H$ which is a contradiction. Thus $n = m$ and so $h' = h$.

Since $G_m \supseteq \bigsqcup_{j=0}^m g_0^j H_{m-j}$, we have

$$\#G_m \geq \sum_{j=0}^m \#g_0^j H_{m-j}.$$

By the definition of polynomial growth, there exists $\alpha \in \mathbb{R}_{>0}$ such that $\#H_m \geq \alpha m^k$ for all $m \in \mathbb{Z}_{\geq 0}$. Thus

$$\sum_{j=0}^m \#g_0^j H_{m-j} = \sum_{j=0}^m \#H_j \geq \alpha \cdot \sum_{j=0}^m j^k.$$

Therefore $\#G_m \geq \alpha \cdot \sum_{j=0}^m j^k$.

Claim 2. There is a positive constant $c \in \mathbb{R}_{>0}$ such that $\sum_{j=0}^m j^k \geq c \cdot m^{k+1}$ for all $m \in \mathbb{Z}_{\geq 0}$.

Indeed, It is well known that

$$\sum_{j=1}^m j^k = \frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} \left((-1)^j \sum_{l=0}^j \frac{1}{l+1} \sum_{n=0}^l (-1)^n \binom{l}{n} n^j \right) m^{k-j+1}$$

For sufficiently large $M \gg k$, we have $\sum_{j=1}^M j^k \approx \frac{1}{k+1} M^{k+1}$. Hence there is a desired positive constant $c \in \mathbb{R}_{>0}$.

By this claim, we have $\#G_m \geq \alpha c \cdot m^{k+1}$ for all $m \in \mathbb{Z}_{\geq 0}$. Therefore the growth of G is more than k. This contradicts with the hypothesis.

3. A Key lemma and the proof of the main theorem

The following key lemma will complete the proof of main theorem.

Lemma 3.1. *Let $\mathcal{F}$ a codimension one transversely orientable minimal C^0 foliation on a manifold M. Suppose that each fundamental group $\pi_1(L)$ of any leaf L has polynomial growth of degree k for some nonnegative integer k. If there is a leaf $L_0 \in \mathcal{F}$ without holonomy such that the induced homomorphism $i_* : \pi_1(L_0) \to \pi_1(M)$ of the inclusion $i : L_0 \to M$ is injective, then $\mathcal{F}$ is without holonomy.*

Proof. Let $F : \bigvee_k S^1 \times [0,1] \to M$ be a positive trivial fence on a leaf L_0, $i_t : L_t \to M$ inclusions, $F_t : \bigvee_k S^1 \to L_t$ the induced maps of $F(\cdot, t)$ with $i_t \circ F_t = F(\cdot, t)$, and $H_t := \mathrm{im}(F_{t*}) \leq \pi_1(L_t)$ subgroups for any $t \in [0,1]$ such that $H_0 = \pi_1(L_0)$. Denote by $\mathrm{gr}(G)$ the degree of the polynomial growth of a group G. Then $\mathrm{gr}(\pi_1(L_0)) = \mathrm{gr}(i_{0*}(\mathrm{im}(F_{0*}))) = \mathrm{gr}(i_{t*}(\mathrm{im}(F_{t*}))) \leq \mathrm{gr}(H_t)$ for any $t \in [0,1]$. Since the fundamental groups of all leaves have polynomial growth of degree k, we obtain that $\mathrm{gr}(H_t) \leq \mathrm{gr}(\pi_1(L_t)) = \mathrm{gr}(\pi_1(L_0)) = \mathrm{gr}(H_t)$ and thus that H_t and $\pi_1(L_t)$ have the same polynomial growth. By Lemma 2.4, we have that for any $t \in [0,1]$ and any $g \in \pi_1(L_t)$, there exists some $n \in \mathbb{Z}_{>0}$ such that $g^n \in \mathrm{im}F_{t*}$. Since any codimension one transversely orientable foliation has no finite holonomy, the leaves in the saturation of $F(\bigvee_k S^1 \times]0,1[)$ are without holonomy. By the minimality of $\mathcal{F}$, we obtain the saturation is M. Thus $\mathcal{F}$ is without holonomy.

Now we prove the main theorem.

Proof of Theorem1.1. Since the union of the leaves without holonomy is dense G_δ [EMT], there is a leaf L of $\mathcal{F}$ without holonomy. Since $\mathcal{F}$ has no vanishing cycles, by Theorem 3.4. p.147 [HH], we obtain that the homomorphism $\pi_1(L) \to \pi_1(M)$ induced by the inclusion $L \to M$ is injective. By Lemma 3.1, we have $\mathcal{F}$ is without holonomy.

We have the following corollary.

Corollary 3.2. *Let $\mathcal{F}$ a codimension one transversely orientable transversely real-analytic minimal foliation of a manifold M. If the fundamental group $\pi_1(L)$ of each leaf L has polynomial growth of degree k for some nonnegative integer k, then the foliation $\mathcal{F}$ is without holonomy.*

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